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GENERALIZED DISPLACEMENTS AND THE ACCURACY OF CLASSICAL PLATE THEORY

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Abstract—This paper investigates, by means of statically/kinematically admissible constructions and the hypersphere theorem, the validity of classical (Kirchhoff) plate theory as an approximation to linear elasticity theory. Two levels of accuracy are shown to lead to two entirely different interpretations of admissible generalized displacements of classical plate theory. At lower accuracy, a wealth of admissible interpretations is disclosed, leading to extreme flexibility in the modeling of geometric boundary conditions. At higher accuracy, that flexibility is lost and only novel generalized displacements, describing lateral deflection and rotations of the faces of the plate, are demonstrated to be admissible. To this end, a refined Kirchhoff hypothesis is formulated which avoids all of the notorious inconsistencies of the standard Kirchhoff hypothesis. At both levels of accuracy, the effect of large transverse-shear deformability is exposed, to encompass anisotropic and composite plates.

1. INTRODUCTION

The classical (Kirchhoff) plate theory (CPT) has been the subject of theoretical and practical interest for over a century. Its mathematical structure is well understood. Its derivation by descent from 3-D elasticity theory is achieved either variationally or asymptotically, in the limit of vanishing plate thickness. However, in real plates, with *finite* thickness, the potential of CPT as a 2-D theory for furnishing adequate 3-D displacement and stress distributions is, in our view, not fully explored and understood. We plan to show that, because of its approximate character, CPT may be interpreted in a variety of ways, depending on the accuracy desired. To prove this point, we start from the 2-D equations of CPT and construct 3-D statically and kinematically admissible solutions. The error of these solutions with respect to (unknown) elasticity solutions is then evaluated in an energy norm, on the basis of the hypersphere theorem of Prager and Synge (1947). This approach was first applied to CPT by Nordgren (1971) who found the error to be $O(h)$, $2h$ being the thickness of the plate. Then Simmonds (1971) and Nordgren (1972) reduced the error to $O(h^2)$.

This work demonstrates that each of the two estimates, $O(h)$, $O(h^2)$, is valid in its own right: their validity domains overlap but are not identical. These domains are found to be defined by the range of admissible interpretations of generalized displacements—transverse displacement and cross-section rotations—of CPT. We show that a decrease in the error of CPT tightens the admissibility range. In particular, at the $O(h)$ level of error, we generalize Nordgren's (1971) 3-D kinematically admissible displacement field, based on the so-called modified Kirchhoff hypothesis due to Koiter (1970), to a one-parameter family of fields. Here, an infinite number of interpretations for generalized displacements are found to be admissible, including all known versions such as midplane or average (through-the-thickness) displacement and rotations. Accordingly, geometric boundary conditions can be modeled in a great variety of ways in that (low accuracy) version of CPT.

At the $O(h^2)$ level of error, most of the generalized displacements from the previous level become inadmissible. Specifically, both midplane-associated (classical) and average displacements and rotations are left out. At this level, we use a refined Kirchhoff hypothesis to get rid of all the notorious contradictions of the conventional (and modified) Kirchhoff hypothesis, so that transverse shear and normal strains are accounted for. The refined hypothesis selects displacements and rotations of the top and bottom surfaces of the plate

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as the only admissible (pointwise) generalized displacements, an apparently physically sound and novel choice for CPT.

In all our developments we investigate and expose the effect of increased transverse-shear deformability, which is frequent in anisotropic and composite plates. This effect is shown to impair the accuracy of 3-D solutions based on CPT, being stronger in the $O(h^2)$ than the $O(h)$ region.

2. ELASTICITY PROBLEM STATEMENT

Within linear elasticity, consider a plate of constant thickness, which is in static equilibrium without body forces,

$$\sigma_{\alpha\beta,\beta}+\sigma_{\alpha 3,3}=0, \quad \sigma_{\alpha 3,\alpha}+\sigma_{33,3}=0, \tag{1}$$

where $\boldsymbol{\sigma}(x^\lambda, z)$ is the stress tensor, x^λ denotes arbitrary in-plane coordinates, $z \equiv x^3$ measures distance from the middle plane ($z=0$), Greek indices range over 1, 2 and are summed when repeated, $(\ )_{,\alpha} \equiv \partial(\ )/\partial x^\alpha$ and $(\ )_{,3} \equiv \partial(\ )/\partial x^3$. The constitutive equations, for a homogeneous, elastic, anisotropic material having symmetry with respect to planes z = constant, read

$$\sigma_{\alpha\beta}=D_{\alpha\beta\lambda\eta}u_{\lambda,\eta}+C_{\alpha\beta}\sigma_{33}, \quad \sigma_{\alpha 3}=E_{\alpha 3\beta 3}(u_{\beta,3}+u_{3,\beta}), \tag{2a, b}$$

$$\sigma_{33}=E_{3333}(u_{3,3}+C_{\alpha\beta}u_{\alpha,\beta}), \tag{2c}$$

where $\mathbf{u}(x^\lambda, z)$ is the displacement vector, $\mathbf{E}$ the elasticity tensor. As a standard preparation for using plane stress approximation, the transverse normal strain $u_{3,3}$ has been eliminated from eqn (2a), giving rise to two tensors:

$$D_{\alpha\beta\lambda\eta}=E_{\alpha\beta\lambda\eta}-E_{\alpha\beta 33}E_{33\lambda\eta}/E_{3333}, \quad C_{\alpha\beta}=E_{\alpha\beta 33}/E_{3333}. \tag{3}$$

The top and bottom surfaces share in equal parts a distributed normal load $p(x^\lambda)$, without tangential tractions,

$$\sigma_{\alpha 3}=0, \quad \sigma_{33}=\pm(1/2)p \quad \text{for } z=\pm h, \tag{4}$$

$2h$ being the thickness of the plate. Part S_σ of the cylindrical edge surface S is subject to the arbitrary forces $f^*_\alpha(s, z)$, $f^*_3(s, z)$, odd and even in z,

$$\sigma_{\alpha\beta}n_\beta=f^*_\alpha, \quad \sigma_{\alpha 3}n_\alpha=f^*_3 \quad \text{on } S_\sigma, \tag{5}$$

where s runs along the edge of the midplane and n_α is a unit normal to S. The remainder, S_u, of S is displaced by $u^*_\alpha(s, z)$, odd in z, and $u^*_3(s, z)$, even,

$$u_\alpha=u^*_\alpha, \quad u_3=u^*_3 \quad \text{on } S_u. \tag{6}$$

The solution to eqns (1)–(6) is split into $(\sigma_{\alpha\beta}, \sigma_{33}, u_\alpha)$, which fields are odd in z, and $(\sigma_{\alpha 3}, u_3)$, which are even. We will seek two-fold approximate solutions: a statically admissible stress field $\tilde{\boldsymbol{\sigma}}(x^\lambda, z)$, which fulfills only eqns (1), (4), (5)—which do not involve displacements, and a kinematically admissible displacement $\hat{\mathbf{u}}(x^\lambda, z)$, which conforms to the geometric boundary conditions (6) and produces, through eqns (2), the stress field $\hat{\boldsymbol{\sigma}}(\hat{\mathbf{u}})$. The distance of $(\tilde{\boldsymbol{\sigma}}, \hat{\mathbf{u}})$ from the (unknown) exact solution $(\boldsymbol{\sigma}, \mathbf{u})$ is bounded by the inequalities

$$\|\tilde{\boldsymbol{\sigma}}-\boldsymbol{\sigma}\| \leqslant \|\tilde{\boldsymbol{\sigma}}-\hat{\boldsymbol{\sigma}}\|, \quad \|\hat{\mathbf{u}}-\mathbf{u}\| \leqslant \|\tilde{\boldsymbol{\sigma}}-\hat{\boldsymbol{\sigma}}\|, \tag{7}$$

which follow from the familiar hypersphere theorem due to Prager and Synge (1947). The norms $\|\boldsymbol{\sigma}\|$ and $\|\mathbf{u}\| \equiv \|\boldsymbol{\sigma}(\mathbf{u})\|$ are based on the positive-definite energy functional

$$\|\sigma\|^2 = \int_F \int_{-h}^{h} (A_{\alpha\beta\lambda\eta}\sigma_{\alpha\beta}\sigma_{\lambda\eta} + 2A_{\alpha\beta 33}\sigma_{\alpha\beta}\sigma_{33} + 4A_{\alpha 3\beta 3}\sigma_{\alpha 3}\sigma_{\beta 3} + A_{3333}\sigma_{33}\sigma_{33})\, \mathrm{d}F\, \mathrm{d}z, \quad (8)$$

where **A** is the compliance tensor (inverse of **E**) and F is the middle plane.

To concisely expose the effect of transverse-shear deformability, the material tensors are scaled as

$$(E_{\alpha\beta\lambda\eta}, E_{3333}, E_{\alpha\beta 33}, D_{\alpha\beta\lambda\eta}) = E(\bar{E}_{\alpha\beta\lambda\eta}, \bar{E}_{3333}, \bar{E}_{\alpha\beta 33}, \bar{D}_{\alpha\beta\lambda\eta}), \quad (9a)$$

$$(A_{\alpha\beta\lambda\eta}, A_{3333}, A_{\alpha\beta 33}) = (1/E)(\bar{A}_{\alpha\beta\lambda\eta}, \bar{A}_{3333}, \bar{A}_{\alpha\beta 33}), \quad (9b)$$

$$E_{\alpha 3\beta 3} = G\bar{E}_{\alpha 3\beta 3}, \quad A_{\alpha 3\beta 3} = (1/G)\bar{A}_{\alpha 3\beta 3}, \quad (9c)$$

where E, G are chosen to make the barred tensors dimensionless and $O(1)$. One may think of E and G as a generalized in-plane Young's modulus and generalized transverse-shear modulus. In well-designed anisotropic and composite plates, typically $E \gg G$.

3. CLASSICAL PLATE THEORY

Classical plate theory consists of the gross equilibrium equations

$$M_{\alpha\beta,\beta} = Q_\alpha, \quad Q_{\alpha,\alpha} = -p, \quad (10)$$

the constitutive relations

$$M_{\alpha\beta} = -(2/3)h^3 D_{\alpha\beta\lambda\eta} w_{,\lambda\eta}, \quad (11)$$

the static boundary conditions

$$M_{\alpha\beta}n_\alpha n_\beta = M^*, \quad (M_{\alpha\beta}n_\alpha t_\beta)_{,\lambda}t_\lambda + Q_\alpha n_\alpha = H^* \quad \text{on } s_\sigma, \quad (12)$$

and the kinematic boundary conditions

$$w = w^*, \quad w_{,\alpha}n_\alpha = r^* \quad \text{on } s_u. \quad (13)$$

$M^*(s)$ and $H^*(s)$ are a prescribed bending moment and generalized transverse force on the edge,

$$M^*(s) = \int_{-h}^{h} n_\alpha f_\alpha^*(s,z) z\, \mathrm{d}z, \quad H^*(s) = \int_{-h}^{h} \{z[t_\alpha f_\alpha^*(s,z)]_{,\lambda}t_\lambda + f_3^*(s,z)\}\, \mathrm{d}z, \quad (14)$$

$w^*(s)$ and $r^*(s)$ are prescribed generalized or gross measures of transverse displacement and rotation about the tangent to the edge,

$$w^*(s) = \int_{-h}^{h} W_3(z) u_3^*(s,z)\, \mathrm{d}z, \quad r^*(s) = \int_{-h}^{h} W(z) n_\alpha u_\alpha^*(s,z)\, \mathrm{d}z, \quad (15)$$

where s_σ and s_u denote the intersections of S_σ and S_u with the edge of the middle plane, t_λ stands for a unit tangent to the edge, $W_3(z)$ is an even function, and $W(z)$ is an odd function.

Equations (14) and (15) reduce the edge data of the 3-D problem of elasticity to the edge data of CPT. A similar reduction of 3-D statically admissible stresses $\tilde{\sigma}$ and kinematically admissible displacements $\hat{\mathbf{u}}$ to 2-D generalized forces $M_{\alpha\beta}$, Q_α and generalized displacements w, $w_{,\alpha}$ is postulated in the interior of the plate:

$$M_{\alpha\beta}(x^\lambda) = \int_{-h}^{h} \tilde{\sigma}_{\alpha\beta}(x^\lambda, z) z \, dz, \quad Q_\alpha(x^\lambda) = \int_{-h}^{h} \tilde{\sigma}_{\alpha 3}(x^\lambda, z) \, dz, \tag{16}$$

$$w(x^\lambda) = \int_{-h}^{h} W_3(z) \hat{u}_3(x^\lambda, z) \, dz, \quad w_{,\alpha}(x^\lambda) = \int_{-h}^{h} W(z) \hat{u}_\alpha(x^\lambda, z) \, dz. \tag{17}$$

Definitions (14) and (16) of the static quantities of CPT are standard and unique. By contrast, eqns (15) and (17) describe the kinematic quantities in as general terms as possible. Owing to the arbitrary weight functions $W_3(z)$, $W(z)$, the generalized displacements of CPT may be given diverse interpretations which offer, importantly, diverse models of geometric edge constraints. Thus, unlike most works on CPT, we do not *preassign* any particular interpretation of the generalized displacements; rather, admissible interpretations will *result* in Sections 4 and 5 from error analyses of CPT as an approximation to elasticity theory.

4. FAMILY OF LOWER ACCURACY 3-D SOLUTIONS

Our task is to construct $\tilde{\boldsymbol{\sigma}}[w(x^\lambda), z]$, $\hat{\mathbf{u}}[w(x^\lambda), z]$ and investigate the relationship between the error $\|\tilde{\boldsymbol{\sigma}} - \hat{\boldsymbol{\sigma}}(\hat{\mathbf{u}})\|$ of the 3-D approximate solutions $\tilde{\boldsymbol{\sigma}}$, $\hat{\boldsymbol{\sigma}}$ and the range of associated interpretations of 2-D generalized displacements w, $w_{,\alpha}$ of CPT.

We use a standard [see e.g. Nordgren (1971)], statically admissible stress field, with $\zeta \equiv z/h$,

$$\tilde{\sigma}_{\alpha\beta} = (3\zeta/2h^2) M_{\alpha\beta}, \quad \tilde{\sigma}_{\alpha 3} = (3/4h)(1-\zeta^2) Q_\alpha, \quad \tilde{\sigma}_{33} = (1/4)(3\zeta - \zeta^3) p. \tag{18}$$

By eqns (10), this stress satisfies 3-D equilibrium (1) and top/bottom boundary conditions (4). Because of eqns (14) and (16), it fulfills the gross or reduced static conditions (12) on the edge. The pointwise conditions (5) cannot be met within the confines of CPT for arbitrary data forces $f^*_\alpha(z)$, $f^*_3(z)$; practically, this is not much of a drawback as one hardly ever knows the precise through-the-thickness distribution of edge tractions.

Use of eqns (10) in eqns (18) yields the order-of-magnitude estimates

$$\tilde{\sigma}_{\alpha 3} = O(\tilde{\sigma}_{\alpha\beta} h/l), \quad \tilde{\sigma}_{33} = O(\tilde{\sigma}_{\alpha\beta} h^2/l^2), \tag{19}$$

l being a characteristic wavelength of 2-D solutions. For slowly varying solutions, one gets $h/l \ll 1$ and the state of stress is approximately plane with $\tilde{\sigma}_{\alpha\beta} \gg \tilde{\sigma}_{\alpha 3} \gg \tilde{\sigma}_{33}$. Thus, the simplest reasonable displacement $\hat{\mathbf{u}}$ is one that produces a stress $\hat{\sigma}_{\alpha\beta}$ close to $\tilde{\sigma}_{\alpha\beta}$ and larger than $\hat{\sigma}_{\alpha 3}$, $\hat{\sigma}_{33}$; the details of $\hat{\sigma}_{\alpha 3}$ and $\hat{\sigma}_{33}$ distributions across the thickness are immaterial. Expecting so little of $\hat{\mathbf{u}}$, we may find a one-parameter family of conforming displacements fields

$$\hat{u}_\alpha = -\zeta h w_{,\alpha}, \quad \hat{u}_3 = w + (\zeta^2 - c) g, \tag{20}$$

where c is a parameter, and

$$g = (h^2/2) C_{\alpha\beta} w_{,\alpha\beta}. \tag{21}$$

Substitution of eqns (20) and (21) into eqns (2) and use of eqn (11) gives

$$\hat{\sigma}_{\alpha\beta} = (3\zeta/2h^2) M_{\alpha\beta}, \quad \hat{\sigma}_{\alpha 3} = (\zeta^2 - c) E_{\alpha 3\beta 3} g_{,\beta}, \quad \hat{\sigma}_{33} = 0. \tag{22}$$

Subtracting eqns (22) from eqns (18) we find

$$\tilde{\sigma}_{\alpha\beta} - \hat{\sigma}_{\alpha\beta} = 0, \quad \tilde{\sigma}_{33} - \hat{\sigma}_{33} = (1/4)(3\zeta - \zeta^3) p, \tag{23a, b}$$

$$\tilde{\sigma}_{\alpha 3} - \hat{\sigma}_{\alpha 3} = (3/4h)(1-\zeta^2) Q_\alpha + (c - \zeta^2) E_{\alpha 3\beta 3} g_{,\beta}. \tag{23c}$$

At this point we restrict the parameter c to $0 \leqslant c \leqslant 1$; otherwise, the right-hand side of eqn (23c) could become arbitrarily large. Thanks to eqns (10), (11) and (21) all x^λ-dependent fields in eqns (18), (22) and (23) can be expressed through the single variable $w(x^\lambda)$ of CPT. Performing this, introducing the results into eqns (7), using eqns (9), and leaving only lowest-order terms in h, yields

$$(\|\tilde{\boldsymbol{\sigma}}-\boldsymbol{\sigma}\|/\|\tilde{\boldsymbol{\sigma}}\|, \quad \|\hat{\mathbf{u}}-\mathbf{u}\|/\|\hat{\mathbf{u}}\|) \leqslant (E/G)^{1/2}h/L+O(h^2), \tag{24}$$

where L is a constant, having the dimension of length, that captures (integrally) the dependence of eqn (24) on w; we will refer to L as a global characteristic wavelength of CPT. An explicit formula for L is complex and unnecessary since the relative errors estimated parametrically in eqn (24) are directly computable from eqns (7) once w (and so $\tilde{\boldsymbol{\sigma}}$, $\hat{\boldsymbol{\sigma}}$) is known.

Introducing eqn (20b) into (17a) gives

$$\int_{-1}^{1} W_3(\zeta)\,\mathrm{d}\zeta = 1, \quad \int_{-1}^{1} W_3(\zeta)(\zeta^2-c) = 0, \quad 0 \leqslant c \leqslant 1. \tag{25}$$

There are infinitely many pairs $\{W_3(\zeta), c\}$ in accord with eqns (25) and so, by eqn (17a), there are just as many admissible interpretations of w as a measure of transverse displacement. Three familiar choices are

$$w(x^\lambda) = \hat{u}_3(x^\lambda, 0), \quad c = 0, \tag{26a}$$

$$w(x^\lambda) = (1/2)\int_{-1}^{1} \hat{u}_3(x^\lambda, \zeta)\,\mathrm{d}\zeta, \quad c = 1/3, \tag{26b}$$

$$w(x^\lambda) = (3/4)\int_{-1}^{1} (1-\zeta^2)\hat{u}_3(x^\lambda, \zeta)\,\mathrm{d}\zeta, \quad c = 1/5, \tag{26c}$$

where w represents midsurface displacement (most popular, historically first choice), average through-the-thickness displacement, and weighted displacement proposed by Reissner (1944) in his higher-order (shear-deformation) theory of plates, respectively. Equations (26) may be viewed as particular cases of the following one-parameter families:

$$w(x^\lambda) = (1/2)[\hat{u}_3(x^\lambda, \zeta_0)+\hat{u}_3(x^\lambda, -\zeta_0)], \quad 0 \leqslant \zeta_0 = c^{1/2} \leqslant 1, \tag{27}$$

$$w(x^\lambda) = \int_{-1}^{1} (a-b\zeta^2)\hat{u}_3(x^\lambda, \zeta)\,\mathrm{d}\zeta, \tag{28}$$

where

$$a = \frac{3}{6-2b/a}, \quad b = \frac{3}{6a/b-2}, \quad \frac{a}{b} = \frac{3-5c}{5-15c}, \quad 0 \leqslant c \leqslant 1. \tag{29}$$

Equation (27) provides pointwise interpretations of w at two equidistant planes $\zeta = \pm\zeta_0$, beginning from the midsurface ($\zeta_0 = 0$) and ending at the top/bottom surfaces ($\zeta_0 = \pm 1$). Equations (28) and (29) contain average interpretations with second-degree polynomial weight functions. Higher-order polynomials may also be considered.

Substitution of eqn (20a) into eqn (17b) yields

$$-h\int_{-1}^{1} W(\zeta)\zeta\,d\zeta = 1. \tag{30}$$

Equation (30) is satisfied by an infinity of weight functions $W(\zeta)$; this leads via eqn (17b) to infinitely many measures $w_{,\alpha}$ of cross-section rotation. Two familiar examples are

$$w_{,\alpha}(x^\lambda) = -\hat{u}_{\alpha,3}(x^\lambda, 0), \quad w_{,\alpha}(x^\lambda) = -(3/2h)\int_{-1}^{1} \hat{u}_\alpha(x^\lambda, \zeta)\zeta\,d\zeta, \tag{31}$$

where $w_{,\alpha}$ represents local rotation at the midsurface (classical choice) and average rotation, respectively. The local interpretation (31a) is easily extended to a family of local rotations at two equidistant planes $\zeta = \pm\zeta_0$,

$$w_{,\alpha}(x^\lambda) = -(1/2)[\hat{u}_{\alpha,3}(x^\lambda, \zeta_0)+\hat{u}_{\alpha,3}(x^\lambda, -\zeta_0)], \quad 0 \leqslant \zeta_0 \leqslant 1. \tag{32}$$

In another family,

$$w_{,\alpha}(x^\lambda) = -(1/2h\zeta_0)[\hat{u}_\alpha(x^\lambda, \zeta_0)-\hat{u}_\alpha(x^\lambda, -\zeta_0)], \quad 0 < \zeta_0 \leqslant 1, \tag{33}$$

rotation is measured as the difference between in-plane displacements of two equidistant planes $\zeta = \pm\zeta_0$. The special case involving top/bottom surfaces ($\zeta_0 = \pm 1$) was discussed by Rehfield and Murthy (1982) in the context of refined beam theory.

The findings of this section are novel in several respects. First, we have come out with a one-parameter family of 3-D displacement fields in eqns (20), thus generalizing the so-called modified Kirchhoff hypothesis (corresponding to $c = 0$) due to Koiter (1970) and Nordgren (1971). Consequently, our error estimate (24) generalizes Nordgren's (1971) similar result to all possible versions of CPT which differ from each other by their generalized displacements. The range of admissible interpretations for these displacements has been identified. We have shown that one is offered an extreme flexibility in the choice of generalized displacements, with several admissible families of transverse displacements w and rotations $w_{,\alpha}$. Representatives of these families can enter each specific pair (w, $w_{,\alpha}$) in any combination, thus providing diverse models of geometric boundary conditions. Note that while $w_{,\alpha}$ is *numerically* related to w as its derivative, *physically* they need not be linked. One can go far beyond the historically first, and most popular to this day, selection of w and $w_{,\alpha}$ as midplane displacement and rotation. For example, w may measure top surface deflection and $w_{,\alpha}$ midsurface rotation or w may be taken to be an average displacement and $w_{,\alpha}$ a pointwise rotation. In a concrete situation, one should select an interpretation that best fits the geometric edge constraints at hand. This is the most one can expect of CPT which, as a 2-D theory, cannot in general fulfill 3-D elasticity geometric boundary conditions (6) at all points across the thickness.

5. HIGHER ACCURACY 3-D SOLUTIONS

Equations (18), (19), (22) and (23) indicate that for better accuracy it is necessary to improve on $\hat{\sigma}_{\alpha 3}$, making it close to $\tilde{\sigma}_{\alpha 3}$. This is achieved by taking

$$\hat{u}_\alpha = -\zeta h w_{,\alpha}+(\zeta^3-3\zeta)b_\alpha, \quad \hat{u}_3 = w+(\zeta^2-1)g, \tag{34}$$

where

$$g = (h^2/2)C_{\alpha\beta}w_{,\alpha\beta}, \quad b_\alpha = -(h/3)g_{,\alpha}-A_{\alpha 3\beta 3}Q_\beta. \tag{35}$$

Introduction of eqns (34) and (35) into (2), use of eqn (11) and of the familiar relation

$$4E_{\alpha 3\beta 3}A_{\beta 3\lambda 3} = \delta_{\alpha\lambda},$$

where $\delta_{\alpha\lambda}$ is the Kronecker delta, produces

$$\hat{\sigma}_{\alpha\beta} = (3\zeta/2h^2)M_{\alpha\beta} + (\zeta^3 - 3\zeta)(D_{\alpha\beta\lambda\eta} + E_{3333}C_{\alpha\beta}C_{\lambda\eta})b_{\lambda,\eta}, \tag{36a}$$

$$\hat{\sigma}_{33} = (\zeta^3 - 3\zeta)E_{3333}C_{\alpha\beta}b_{\alpha,\beta}, \tag{36b}$$

$$\hat{\sigma}_{\alpha 3} = (3/4h)(1-\zeta^2)Q_{\alpha}. \tag{36c}$$

Subtraction of eqns (36) from eqns (18) yields

$$\tilde{\sigma}_{\alpha\beta} - \hat{\sigma}_{\alpha\beta} = (3\zeta - \zeta^3)(D_{\alpha\beta\lambda\eta} + E_{3333}C_{\alpha\beta}C_{\lambda\eta})b_{\lambda,\eta}, \tag{37a}$$

$$\tilde{\sigma}_{33} - \hat{\sigma}_{33} = (3\zeta - \zeta^3)(p/4 + E_{3333}C_{\alpha\beta}b_{\alpha,\beta}), \tag{37b}$$

$$\tilde{\sigma}_{\alpha 3} - \hat{\sigma}_{\alpha 3} = 0. \tag{37c}$$

The 3-D stresses in eqns (18), (36) and (37) can be expressed in terms of a 2-D solution w of CPT by means of eqns (10), (11) and (35). Performing this, substituting the results into eqns (7), using eqns (9), and leaving only lowest-order terms in h gives the improved error estimate for CPT:

$$(\|\tilde{\boldsymbol{\sigma}} - \boldsymbol{\sigma}\|/\|\tilde{\boldsymbol{\sigma}}\|, \quad \|\hat{\mathbf{u}} - \mathbf{u}\|/\|\hat{\mathbf{u}}\|) \leqslant (E/G)h^2/L_*^2 + O(h^3), \tag{38}$$

where L_* is a refined global wavelength of CPT; it has a similar meaning to L but depends on w differently.

Equations (17) and (34) yield

$$\int_{-1}^{1} W_3(\zeta)\,d\zeta = 1, \quad \int_{-1}^{1} W_3(\zeta)(\zeta^2 - 1)\,d\zeta = 0, \tag{39}$$

$$-h\int_{-1}^{1} W(\zeta)\zeta\,d\zeta = 1, \quad \int_{-1}^{1} W(\zeta)(\zeta^3 - 3\zeta)\,d\zeta = 0. \tag{40}$$

The weight functions W_3, W which fit eqns (39) and (40) determine admissible interpretations of generalized displacements. Looking for pointwise interpretations, we find

$$w(x^{\lambda}) = (1/2)[\hat{u}_3(x^{\lambda}, 1) + \hat{u}_3(x^{\lambda}, -1)], \tag{41a}$$

$$w_{,\alpha}(x^{\lambda}) = -(1/2)[\hat{u}_{\alpha,3}(x^{\lambda}, 1) + \hat{u}_{\alpha,3}(x^{\lambda}, -1)]. \tag{41b}$$

Equations (41) describe w and $w_{,\alpha}$ as average top/bottom surface displacement and rotations. Such variables are physically sound because edge constraints are often imposed on the faces. Apparently, this type of generalized displacements have never been used in association with CPT [in a higher-order theory of plates, Jemielita (1975) used the same w as we do, but in association with an average rotation]. All familiar interpretations of w and $w_{,\alpha}$ fail to comply with eqns (39) and (40) and are not admissible, to name: middle plane displacement and rotations (classical, Kirchhoff's choice), average displacement and rotations, or weighted-average displacement of the Reissner type. In that respect this report completes earlier papers on the accuracy of CPT [by Simmonds (1971) and Nordgren (1972)], where the issue of admissible generalized displacement was not addressed. We have also identified the influence of increased transverse-shear deformability, $E/G \gg 1$, in our error formulae (38), an effect of importance in anisotropic and composite plates. This effect is more pronounced at the higher level of accuracy, eqn (38), than at the lower level, eqn (24). Contrary to what has been generally believed so far, the better estimate in eqn (38) does not relegate its predecessor (24) because these estimates are associated with two different versions of CPT: both versions have the same *mathematical* structure, eqns (10)–(13), but *physically* different generalized displacements and kinematic boundary conditions. The advantage of higher accuracy in one version is offset by the availability of versatile generalized displacements in the other.

The displacement out field (34) states what may be called a refined Kirchhoff hypothesis: straight lines normal to the undeformed top/bottom surfaces move to bent lines normal to the deformed surfaces; the normals are also stretched/contracted but leave the initial distance between the faces unchanged. This statement is free from all the notorious contradictions of the standard ($b_\alpha = g = 0$) and modified ($b_\alpha = 0$) Kirchhoff hypotheses: it does not contradict plane stress and adequately predicts the parabolic distribution of transverse shears. The refined hypothesis, in other words, shifts the standard hypothesis from the wrong place—the middle surface, where transverse shears reach a maximum, to the right place—the faces, where there is no shear deformation by definition (zero prescribed tangential tractions).

Under specific circumstances, CPT is capable of higher accuracy than in eqns (38). For example, Duva and Simmonds' (1990) work on beams implies that CPT is as accurate as we please in the cylindrical bending of a cantilevered plate strip. However in general, the estimate (38) can only be improved upon by shifting from CPT to higher-order plate theories, with additional degrees of freedom. For example, Berdichevski (1973) and Rychter (1988) have found the error of Reissner theory to be $O(h^3)$.

6. CONCLUSIONS

This work furthers the understanding of the classical or Kirchhoff theory of elastic plates in several ways. Most importantly, it puts the theory in dual perspectives in which the theory exhibits complementary advantages of greater flexibility in the choice of generalized displacements (and in the modeling of kinematic boundary conditions) or of increased accuracy. At lower accuracy, the paper expands the validity of existing error estimates for classical theory to an infinity of variants with diverse generalized displacements. At higher accuracy, our work provides a refined Kirchhoff hypothesis which avoids all contradictions of previous hypotheses and makes the displacements and rotations of the top and bottom surfaces of the plate represent the best and unique pointwise choice of generalized displacements. At both levels of accuracy, the impact of increased transverse-shear deformability, important in anisotropic and composite plates, is exposed.

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